

Magnetic cloaking by a paramagnet/superconductor cylindrical tube in the critical state

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Cloaking of static magnetic fields by a finite thickness type-II superconductor tube being in the full critical state and surrounded by a coaxial paramagnet shell is studied. On the basis of exact solutions to the Maxwell equations, it is shown that, additionally to previous studies assuming the Meissner state of the superconductor constituent, perfect cloaking is still realizable at fields higher than the field of full flux penetration into the superconductor and for arbitrary geometrical parameters of both constituents. It is also proven that simultaneously the structure is fully undetectable under the cloaking conditions. Differently from the case of the Meissner state the cloaking properties in the application relevant critical state are realized, however, only at a certain field magnitude.

A great research activity has been focused last time on electromagnetic metamaterials which exhibit many unique features, providing particularly cloaking of electromagnetic waves as well as of low frequency and static magnetic or electric fields^{1–8}. A magnetic cloak is expected to retain undistorted the external field outside the cloak, thus being “invisible” for external observation, and, as far as possible, to protect its inner area from the external field penetration. To fulfill these requirements, different cloak designs have been proposed, including hybrid systems consisting of ferromagnet and superconductor constituents^{8–13} which were recently experimentally realized in the forms of multilayered¹⁴ or bilayered^{15,16} magnet/superconductor hollow cylinder. An essential component of the proposed hybrid cylindrical designs is the inner superconducting layer which was assumed until quite recently to be an ideal diamagnetic medium with zero effective permeability in both analytical and finite-element considerations^{10,15}. This assumption is, however, unrealistic because of (1) finite field penetration depth which can be comparable with superconductor thickness and (2) massive magnetic flux penetration followed by formation of the critical state typical of magnetic shielding applications^{17–19}.

The system with the finite penetration depth of magnetic field into a superconductor never completely protects the inner region (a central hole) from the penetration of weak external field even in the Meissner state²⁰. On the other hand, a non-distorted uniform magnetic field outside the cloak can exist in a wide range of relative permeability and thickness values of the paramagnet sheath for both cases of thick and thin superconductor layers being in the Meissner state. At the same time, the magnetic moment of such a bilayer tube vanishes under the cloaking conditions (as well as all higher multipole moments) making this object magnetically undetectable. Moreover, penetration of magnetic flux into a superconductor in the form of single vortices produces rather small

paramagnetic moment of the system, thus breaking the perfect cloaking only slightly.

In this respect a following question arises: Is cloaking possible for the case of multiple vortex penetration at higher fields, and particularly when a superconductor is in the critical state as is characteristic of shielding with high-temperature superconductors?^{17–19} In general, comprehensive numerical simulations are necessary to analyze the critical state in structures of such kind, even in simpler geometries like a superconductor filament exposed to transverse magnetic field^{21–26}. However, in the specific case of the superconductor constituent completely penetrated by the magnetic flux, *i.e.* being in the full critical state, the arising problem is analytically solvable.

In the present study, based on the Bean concept of the critical state²⁷, we demonstrate by exact solving the Maxwell equations for respective media that the cloaking effect still holds in a realistic cylindrical design of bilayer paramagnet/superconductor tube with finite thicknesses of both superconducting and magnetic constituents, when the superconducting layer is already in the full critical state. We establish the values of constituent parameters necessary for perfect cloaking and prove also the completely vanishing detectability of this object under the cloaking conditions.

Let us consider an infinitely long hollow superconducting cylinder of thickness d_S and radius of a coaxial hole R_0 enveloped in a coaxial cylindrical magnetic sheath of thickness d_M with relative permeability $\mu > 1$. This structure is exposed to an external constant magnetic field \mathbf{H}_0 perpendicular to the cylinder axis z (see Fig. 1). Magnetic flux enters the superconducting constituent through the superconductor/magnet interface and induces shielding currents in the regions of flux penetration. According to the Bean concept of the critical state, the flux-penetrated regions carry a current of density j_c directed parallel (antiparallel) to the z -direction. Generally, these regions evolve in a rather complicated manner^{21–24,28} but, when magnetic flux completely penetrates the superconductor, the boundary between the re-

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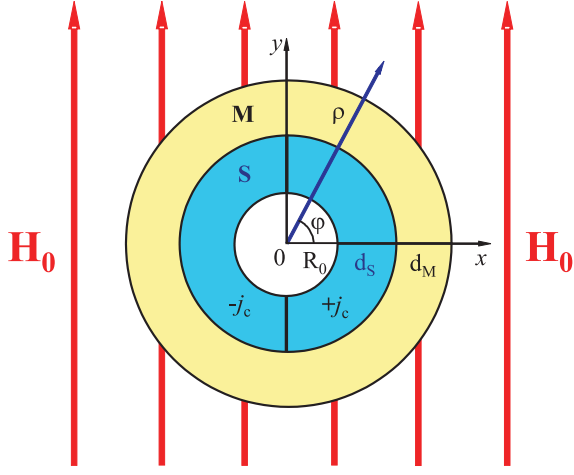


FIG. 1. (color online) Cross-sectional view of a hollow superconductor cylinder in the full critical state, covered by a coaxial cylindrical magnetic sheath and assumed to be in the cloaking regime. The vertical solid lines $x = 0$ denote the boundaries between the superconductor regions with negative and positive directions of the critical current. The direction of the applied magnetic field \mathbf{H}_0 is also indicated.

regions of positive and negative induced currents coincides with the $x = 0$ plane of Cartesian coordinate system, as marked in Fig. 1.

In this case, the magnetic field in the system, denoted by \mathbf{H}_{in} inside the hole, by \mathbf{H}_S in a superconductor, by \mathbf{H}_M in a magnetic sheath and by \mathbf{H}_{out} in a surrounding free space, obey the Maxwell equations²³

$$\text{curl } \mathbf{H} = \mathbf{j}_c [\theta(\rho - R_0) - \theta(\rho - R_1)], \quad \text{div } \mathbf{B} = 0, \quad (1)$$

with $\mathbf{j}_c = j_c \text{sgn}(x) \mathbf{e}_z$, where \mathbf{e}_z is a unit vector in the z -direction, $\text{sgn}(\dots)$ denotes the sign function, $\theta(\dots)$ denotes the Heaviside unit step function, and $R_1 = R_0 + d_S$, cylindrical coordinates (ρ, φ, z) coaxial with the tube being introduced here for convenience. Implying an insulating, nonmagnetic layer of thickness much less than d_M and d_S between the superconductor and the magnet sheath, which is typical for hybrid magnet/superconductor structures (see, for example, Refs.^{29,30}), the boundary conditions read²⁰

$$B_{S,n} = \mu_0 H_{in,n}, \quad B_{S,t} = \mu_0 H_{in,t}; \quad (2a)$$

$$B_{S,n} = \mu_0 \mu H_{M,n}, \quad B_{S,t} = \mu_0 H_{M,t}; \quad (2b)$$

$$\mu H_{M,n} = H_{out,n}, \quad H_{M,t} = H_{out,t}, \quad (2c)$$

for the normal (n) and tangential (t) components on the inner superconductor surface [Eq. (2a)], on the superconductor/magnet interface [Eq. (2b)] and on the outer magnet surface [Eq. (2c)], respectively (cf. also Refs.^{31,32}), with μ_0 permeability of vacuum. In addition, the magnetic field outside the structure has to approach asymptotically the applied field \mathbf{H}_0 .

In cylindrical coordinates (ρ, φ, z) , the solution of

Eqs. (1) takes the form:

$$\begin{aligned} H_{in,\rho} &= A_{in} \sin \varphi, \\ H_{in,\varphi} &= A_{in} \cos \varphi, \end{aligned} \quad (3)$$

in the hole ($\rho \leq R_0$);

$$\begin{aligned} B_{S,\rho} &= \mu_0 (A_{S1}/\rho^2 + A_{S2} + 4j_c\rho/3\pi) \sin \varphi, \\ B_{S,\varphi} &= \mu_0 (-A_{S1}/\rho^2 + A_{S2} + 8j_c\rho/3\pi) \cos \varphi, \end{aligned} \quad (4)$$

in the superconductor ($R_0 \leq \rho \leq R_1$);

$$\begin{aligned} H_{M,\rho} &= (A_{M1}/\rho^2 + A_{M2}) \sin \varphi, \\ H_{M,\varphi} &= (-A_{M1}/\rho^2 + A_{M2}) \cos \varphi, \end{aligned} \quad (5)$$

in the magnet sheath ($R_1 \leq \rho \leq R_2 = R_1 + d_M$); and

$$\begin{aligned} H_{out,\rho} &= (H_0 + A_{out}/\rho^2) \sin \varphi, \\ H_{out,\varphi} &= (H_0 - A_{out}/\rho^2) \cos \varphi, \end{aligned} \quad (6)$$

in the space around the tube ($\rho \geq R_2$). The coefficients A_{in} , A_{S1} , A_{S2} , A_{M1} , A_{M2} and A_{out} determined from the boundary conditions (2) are given by the following expressions

$$\begin{aligned} A_{in} &= 4\mu R_2^2 H_0 / \Delta - (2j_c/3\pi) [3(R_1 - R_0) \\ &\quad + (\mu^2 - 1)(R_2^2 - R_1^2)(R_1^3 - R_0^3)/R_1^2 \Delta], \end{aligned} \quad (7)$$

$$A_{S1} = 2j_c R_0^3 / 3\pi, \quad (8)$$

$$\begin{aligned} A_{S2} &= 4\mu R_2^2 H_0 / \Delta - (2j_c/3\pi) [3R_1 + (\mu^2 - 1) \\ &\quad \times (R_2^2 - R_1^2)(R_1^3 - R_0^3)/R_1^2 \Delta], \end{aligned} \quad (9)$$

$$\begin{aligned} A_{M1} &= -2H_0 R_1^2 R_2^2 (\mu - 1) / \Delta \\ &\quad - (4j_c/3\pi) (\mu + 1) (R_1^3 - R_0^3) R_2^2 / \Delta, \end{aligned} \quad (10)$$

$$\begin{aligned} A_{M2} &= 2H_0 R_2^2 (\mu + 1) / \Delta \\ &\quad + (4j_c/3\pi) (\mu - 1) (R_1^3 - R_0^3) / \Delta, \end{aligned} \quad (11)$$

$$\begin{aligned} A_{out} &= H_0 R_2^2 (\mu^2 - 1) (R_2^2 - R_1^2) / \Delta \\ &\quad - (8j_c/3\pi) \mu (R_1^3 - R_0^3) / \Delta, \end{aligned} \quad (12)$$

with $\Delta = (\mu + 1)^2 R_2^2 - (\mu - 1)^2 R_1^2$.

The obtained expressions (7)-(12) are valid only for applied fields H_0 equal or higher than the field of full flux penetration in the superconducting constituent, H_{fp} . The latter is determined by the condition

$$B_S|_{\rho=R_0} = \mu_0 (A_{S1}/R_0^2 + A_{S2} + 4j_c R_0/3\pi) = 0, \quad (13)$$

and reads

$$\begin{aligned} H_{fp} &= H_{fp}^0 \left[1 + \frac{(\mu - 1)(2\mu - 1)(R_2^2 - R_1^2)}{6\mu R_2^2} \right. \\ &\quad \left. + \frac{(\mu^2 - 1)(R_2^2 - R_1^2) R_0 (R_0 + R_1)}{12\mu R_1^2 R_2^2} \right], \end{aligned} \quad (14)$$

where $H_{fp}^0 = (2j_c/\pi)(R_1 - R_0)$ is the field of full flux penetration in the superconductor constituent for an unshielded superconductor cylinder²³. In the limit of $R_0 \rightarrow 0$, H_{fp} reduces to the known expression for the magnetically shielded superconductor filament²⁴.

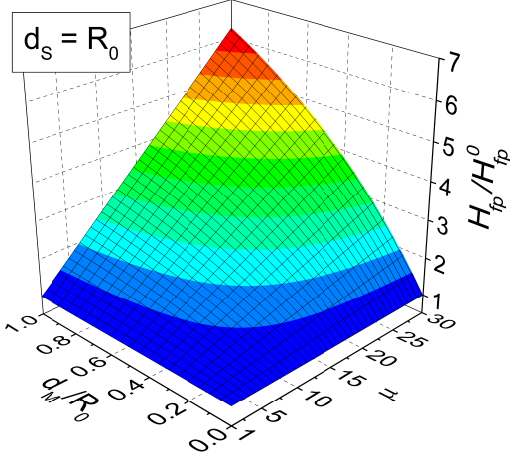


FIG. 2. (color online) The dependence of the field of the full flux penetration H_{fp} on the relative permeability μ and on the thickness d_M of the magnet sheath for the thickness of superconductor layer $d_S = R_0$.

In Fig. 2 we present a typical dependence of the field H_{fp} on the relative permeability μ and on the thickness d_M of the magnet sheath calculated exemplarily for the ratio of superconductor thickness to the inner hole radius $d_S/R_0 = 1$. One can see that due to the shielding of the applied field by the magnetic sheath the field of the full flux penetration generally increases. With the increase of μ at a fixed thickness d_M the $H_{fp}(d_M, \mu)$ dependence approximates to the linear one. A monotonic behavior is also demonstrated by the $H_{fp}(d_M, \mu)$ dependence with the increase of d_M at a fixed permeability μ . For other values of the ratio d_S/R_0 the $H_{fp}(d_M, \mu)$ dependences reveal a qualitatively similar behavior and differ only by the scale of the magnitude. In the limiting case of $d_S \ll R_0$ this field is reduced to

$$H_{fp} = H_{fp}^0 \left[1 + \frac{(\mu - 1)}{2} \frac{d_M (2R_0 + d_M)}{(R_0 + d_M)^2} \right]. \quad (15)$$

Considering the opposite limit $d_S \gg R_0$, closed to the case of the magnetically shielded superconducting filament²⁴, we obtain

$$H_{fp} = H_{fp}^0 \left[1 + \frac{(\mu - 1)(2\mu - 1)}{6\mu} \frac{d_M}{d_S} \right]. \quad (16)$$

In order to retain the magnetic field outside the cloak undisturbed, a condition $A_{out} = 0$ has to be fulfilled. This results in the following equation:

$$\frac{H_{cl}}{H_{fp}^0} = \frac{4}{3} \frac{\mu}{(\mu^2 - 1)} \frac{R_1^2 + R_0 R_1 + R_0^2}{R_2^2 - R_1^2}, \quad (17)$$

from which, for given fixed values of parameters μ, d_M and d_S , a magnitude of applied magnetic field $H_{cl} \geq H_{fp}$ providing the cloaking effect can be found. Notice that

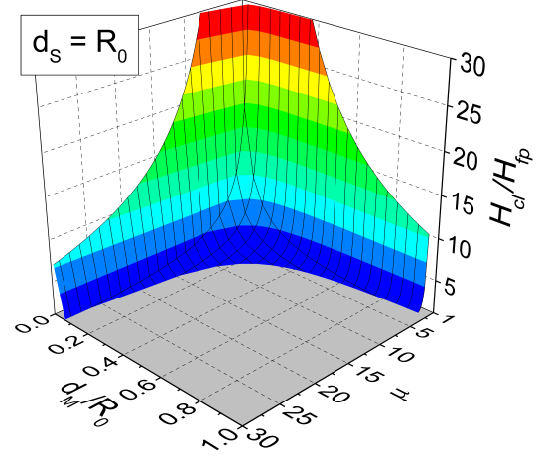


FIG. 3. (color online) The dependence of the ratio H_{cl}/H_{fp} on the relative permeability μ and on the thickness d_M of the magnet sheath for the thickness of superconductor layer $d_S = R_0$.

this is qualitatively different from the case of cloaking in the Meissner state, where the cloaking conditions were field-independent²⁰.

The calculated dependence of H_{cl}/H_{fp} on the relative permeability μ and on the thickness d_M is shown in Fig. 3 for the same value of ratio of d_S/R_0 as in Fig. 2. One can see that the cloaking effect exists for a sufficiently wide range of parameters of the magnet layer restricted by the $\mu(d_M)$ curve corresponding to the solution of Eq. (17) at $H_{cl} = H_{fp}$. This boundary is shown in Fig. 4 for the different values of relative superconductor thickness d_S/R_0 . The $H_{cl}(\mu, d_M)$ surfaces for other d_S/R_0 values are qualitatively similar to that of Fig. 3 differing only by the scale of the magnitude. In the case of rather thin ($d_M \ll R_1$) magnetic sheath the cloaking effect can be

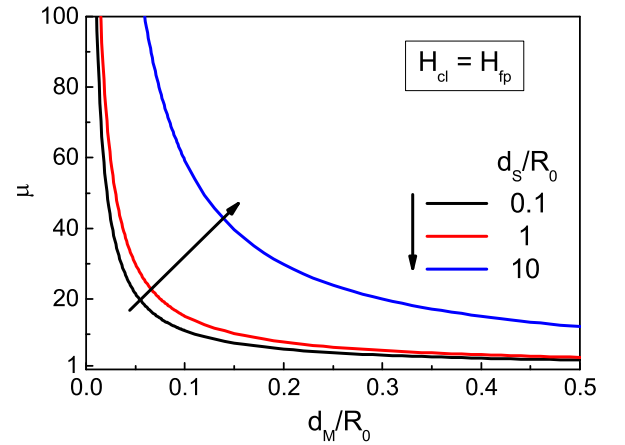


FIG. 4. (color online) Solutions of Eq. (17) with the magnetic field $H_{cl} = H_{fp}$ for different values of the superconductor layer thickness d_S .

provided for both thin ($d_S \ll R_0$) and thick ($d_S \gg R_0$) superconductors if the relative permeability satisfies the condition $\mu d_M/R_1 \simeq H_{fp}^0/H_0$. This is similar to the known effectiveness of magnetic shielding for the Meissner state of the superconducting constituent as soon as the strength of an effective magnetic dipole layer $\mu d_M/R_1$ is notable^{33,34}.

It's interesting to note that also the $\mu(d_M)$ curves in Fig. 4 are qualitatively similar to those obtained earlier²⁰ for cloaking in the case of superconducting constituent being in the Meissner state (cf. Fig. 2 of Ref.²⁰). This hints that the specific cloaking parameters of the structure calculated for both cases – of the Meissner state and of the full critical state – can be related each other. Indeed, using in Eq. (17) the cloaking parameters corresponding to the former case, one can calculate the field H_{cl} . If the obtained value will satisfy the condition $H_{cl} \geq H_{fp}$, then, having in mind possible breaking of cloaking at fields when the magnetic flux starts to penetrate the superconductor²⁰, it would mean that at the field $H_0 = H_{cl}$ cloaking appears again.

As for the possible screening of the inner space of the cloak from the external magnetic field, by using Eq. (14), it follows that

$$A_{in} = (4\mu R_2^2/\Delta) (H_0 - H_{fp}), \quad (18)$$

and, therefore, at field $H_0 = H_{fp}$ the magnetic field inside the cloak vanishes revealing complete protection of the inner hole of the structure from the penetration of magnetic flux. At higher applied fields, a non-zero homogeneous magnetic field $H_{y,in} = A_{in}$ always exists inside the cloak though reduced with respect to the external value.

Another important question concerning the "invisibility" of the coaxial bilayer structure is whether or not it can be detected¹⁰, for example, by measuring a magnetic moment of this structure. Due to the geometry of the system, this moment per unit length of the structure has only a y component and consists of two parts (cf. Refs.^{20,31}): the magnetic moment of the superconductor layer defined by means of the critical current density \mathbf{j}_c as

$$M_S = \int_{V_S} dV [\boldsymbol{\rho} \times \mathbf{j}_c]_y, \quad (19)$$

and the moment of magnetic sheath defined as

$$M_M = (\mu - 1) \int_{V_M} dV H_{M,y}. \quad (20)$$

Using Eqs. (5), (10) and (11), after integration one obtains

$$M_S = -(4j_c/3) (R_1^3 - R_0^3), \quad (21)$$

$$M_M = \pi (\mu - 1) A_{M2} (R_2^2 - R_1^2), \quad (22)$$

and it is easy to ensure that the total magnetic moment $M = M_S + M_M$ per unit length of the paramagnet/superconductor tube being in the superconducting full critical state is expressed through the coefficient

A_{out} ,

$$M = 2\pi A_{out}. \quad (23)$$

This means that in the cloaking case ($A_{out} = 0$, i.e. $H_0 = H_{cl}$) the magnetic moment of the structure under consideration simultaneously vanishes ensuring that the object cannot be detected by magnetic measurements. This is because, according to the form of the solution (3)-(6), the structure does not possess other multipole moments but the dipole one. If the latter equals zero the object cannot be observed by any magnetic measurement, at least as long as an external field uniform at the scale of the object transverse size $\sim R_2$ is involved. Thus, magnetic undetectability of the system happening in the Meissner state can be reestablished at the field H_{cl} .

In conclusion, we have studied theoretically static magnetic cloaking properties of a realistic bilayer paramagnet/superconductor cylindrical tube in the case when the superconducting constituent is completely penetrated by the applied transverse magnetic field. We have found that a non-distorted uniform magnetic field outside the cloak can exist in a wide range of relative permeability and thickness values of the magnet sheath for both cases of thick and thin superconductor layers but only at the specific value of applied magnetic field. Under the above cloaking conditions the magnetic moment of the bilayer structure vanishes (as well as all higher multipole moments) making this object magnetically undetectable. When the applied field equals to the field of full flux penetration, such a system also completely protects the inner region (a central hole) from the penetration of the external field, thus revealing in this case cloaking in its ideal, "dual" form. At higher applied fields, the central hole of the structure is never protected from the magnetic flux penetration. These results are expected to hold also for low frequency ac fields as is generally the case for superconducting shielding¹⁷⁻¹⁹.

Thus, we have demonstrated that, along with a manifestation of both cloaking and complete magnetic undetectability by a paramagnet/superconductor cylindrical structure in the Meissner state known in the literature^{10,11,15,16,20}, these features can be reestablished by increase of an applied magnetic field to a certain value in the range of the superconductor full critical state. The question, whether cloaking (and magnetic undetectability) is possible when the superconductor constituent is only partly penetrated by the magnetic flux, even in the form of a few vortices, needs however a more elaborated treatment.

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